

# Metcalfe's Law Revisited

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**Abstract**—Rudimentary mathematical analysis of simple network models suggests bandwidth-independent saturation of network growth dynamics, as well as hints at linear decrease in information density of the data. However it strongly confirms Metcalfe's law as a measure of network utility and suggests it can play an important role in network calculations.

**Index Terms**—Telecommunications, Networks, Information Systems, Network Effects, Metcalfe's Law

## I. INTRODUCTION

**M**ETCALFE'S law relates to communications networks; it states that the value of a network is proportional to square of its size. Not long ago, a group of authors[1] challenged quadratic dependence. That split community into believers and deniers of Metcalfe's law, and their claim was recently challenged by statistical data[2]. However no attempts were made to establish the truth mathematically, perhaps due to difficulties with obtaining mathematical definition of "value". This paper establishes a notion of value and analyses two conflicting models of network. First, traditional model, fails to manifest Metcalfe's law. Another model, that observes network in a wider context, both confirms Metcalfe's law and shows its upper boundary.

## II. NETWORK VALUE

In lines of Von Neumann–Morgenstern utility theorem[3],

**Definition 1.** Utility of a system is a probability-weighted sum of its value for all possible events:

$$U = \sum_i^{\infty} \phi(K^n, e_i) P(e_i)$$

where  $U_S$  is utility of a system,  $\phi$  is a function on  $n$ -dimensional vector of all system properties  $K_S^n = [k_1, k_2, \dots, k_n]$  given event  $e_i$ , and  $P(e_i)$  is the probability (or relative frequency), of event  $e_i$ .

The definition is universal because it bases value on a scenario. Utility of same system in different scenarios differs (and may be even negative), but it always deterministically follows from system properties.

Function  $\phi$  calculates system utility in case of given event. In order to support claim that network has size-dependent value, we need to show that size-dependent component of  $\phi$  can be separated from event-dependent one. Even when we compare systems that differ in only one parameter, we cannot extract event-independent component because  $a(x, y) = b(x)c(y)$  has no solutions. It shows that network has no universal size-dependent value.

But it is plausible to assume that there exists a non-empty set of events  $E$  for which we can represent  $\phi$  as  $\phi = \xi(e_i)\psi(k)$ , where  $\xi(e_i)$  and  $\psi(k)$  are event-dependent and property-dependent components of  $\phi$ . Note that  $\psi(k)$  is independent of any event, and therefore total utility can be represented as

$$U = \psi(k) \sum_{i=1}^{\infty} \xi(e_i) P(e_i)$$

In this case, if systems differ only in  $k$ , total utility  $U$  can be represented as  $U = \psi(k)C$ , where  $C$  is a system-independent factor. External factors can influence  $C$ , but as long as our system has only one varying parameter, its utility is proportional to a function of that parameter. Now we need to make sure our model meets that criterion.

Information Network is a collection of information consumers and producers connected by communication channels.

In order to make size the only property that distinguishes two networks, we have to add that all nodes and channels are identical, and also that number of channels is a function of network size. Let's also discard constraints by assuming that all parts of a network can process infinite amount of data in no time. Those assumptions are enough to make the notion of network value for set of events  $E$  mathematically consistent. In the next chapters I study and improve this model.

## III. NETWORK EFFECT

Common[4] understanding of network effect is both simple and compelling. The primary function of a network is connecting users. Therefore the value of a network to one user is proportional to the number of other users:  $Q \propto N - 1$  ( $\propto$  denotes proportionality). As a result, total value of a network is proportional to  $N(N-1) = N^2 - N$ . Note that it is identical to the maximum number of unique directed links between nodes.

Let's analyze that claim. By viewing many to many communication as a simultaneous mutual broadcasting, we can model the network as a superposition of broadcast-type networks. See figure 1.

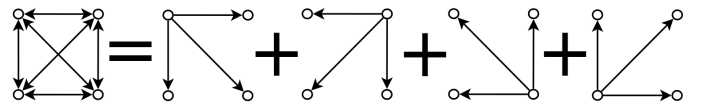


Figure 1. Switched network of size  $N$  can be represented as superposition of  $N - 1$  virtual broadcast networks with  $N - 1$  receivers in each

Network effect basically assumes that those networks are independent. However closer investigation suggests that they are not: though all virtual transmitters indeed are independent, they share same receivers. As a result, links compete for nodes, and each link's share of its terminals equals  $\frac{1}{N-1}$ .

Consequently, in a network with higher density of links one link has proportionally less value. For example, when above network expands  $X$  times, each link's share of its terminals decreases  $X$  times. To illustrate this with telephone networks, potential to place a call for each phone increases, but potential that the call will be answered decreases proportionally. In other words, the gain is imaginary.

One can argue that sharing is applicable only when node capacity is severely constrained. But that is not the case. We need  $N - 1$  links to make a network of size  $N$ , and other links are redundant. That is why net value of a network is always  $N$ . Network effect suggests tendency to exponential growth of partially redundant data, not of value.

**Lemma 2.** *Direct link in unconstrained network has no value.*

*Proof:* In zero-latency, infinite bandwidth network of size  $N$ , node  $X$  has direct connections to every other node. That results in  $N - 1$  direct links. Node  $Y$  has just one connection (of course, to  $X$ ). As a result, we obtain two subnetworks. Network  $A$  has  $N - 1$  nodes (all except  $Y$ ), and network  $B$  has 2 nodes:  $X$  and  $Y$ . Next, we set  $X$  to bridge  $A$  and  $B$ . Now  $Y$  gets exactly the same network service as  $X$ , and the only solution of  $V_{lnk} \times (N - 1) = V_{lnk} \times 1$  is  $V_{lnk} = 0$ . ■

**Corollary 3.** *From 1 follows that all network value is contained in nodes.*

This hints us that information network is identical to a non-distributed information system, such as the computer. In computer memory, you always get same amount of memory from  $N$  units regardless of how you connect those units. Notion that a collection of interlinked nodes can enjoy a non-linear increase in value is mathematically inconsistent, and there are many ways to prove it. For example we can split every network node in  $n$  parts and connect those parts back to the network, which, according to network effect, must raise its value  $\frac{n^2}{n} = n$  times, which is contradiction.

#### IV. NETWORK EFFICACY

Describing network effect as exponential growth of value might originate from difficulties to discriminate redundant data from valuable information. However that neither proves nor disproves Metcalfe's law. The thing is, describing network as a collection of nodes is not the best way to model it. A better point of view is that network has value in previous sense. It provides connectivity between certain parts of (potentially larger), physical or business system, and it is that larger system that has value. Here are the corresponding definitions:

**Definition 4** (Information System). The information system is a collection of complementary subsystems (nodes), that contribute to overall information value.

**Definition 5** (Telecommunications Network). The telecommunications network is an apparatus for information transmission between flexible number of terminals.

According to later definitions, the telecommunications network is just one of possible transmission agents for an information system; it differs from the system itself. As an

illustration, the broadcast network can be viewed as a complex business that delivers its products via telecommunications network. According to the definition, Information System is too complex to derive value from its size. Because of that, information network is a poor approximation of an information system.

To replace value with something more appropriate for a network, let me introduce another definition. It is the efficacy of a network.

**Definition 6** (Network Efficacy). Network Efficacy is the amount of useful data an underlying information system is able to send through a network of  $N$  identical nodes. It is the product of network size and node communication efficacy  $\zeta$ :

$$\psi = N\zeta$$

As before, we investigate an unconstrained network with identical nodes that limits  $\zeta$  neither by bandwidth nor by node capacity. Note that Metcalfe's law holds only when  $\zeta$  is proportional to  $N$ . If  $\zeta$  is constant, network properties are linear in  $N$ . As we will see in a moment,  $\zeta$  exhibits both of those behaviors. Now let me describe a model that reveals this.

#### V. DEFICIT MODEL

Consider the following scenario: User  $X$  wants to contact certain members of her family over Skype. These people can be described by set  $A \subseteq \Omega$ , where  $\Omega$  denotes all nodes of the information system that corresponds to  $X$ 's family.

Skype as a network represents an independent set  $B$ . Those of  $\Omega$  members who use Skype comprise the effective network  $E = B \cap \Omega$ . Note that the actual size of the Skype network is irrelevant Effective network size equals  $|E|$  ( $|\dots|$  denotes cardinality).  $X$  can contact only those people that belong to  $A \cap E$  (the intersection of her contact list and her relatives in Skype network). Expected proportion of people from  $A$  that Skype allows to contact is independent of  $A$ . It is  $\frac{|E|}{|\Omega|}$ , or effective network size divided by information system size. Note that  $|E| \leq |\Omega|$  because  $E \subseteq \Omega$ . The same is true for every member of  $E$ , which leads to the following:

**Theorem 7** (Network Efficacy Theorem). *Network Efficacy is proportional to square of effective network size divided by information system size, where effective network represents the networked part of the information system.*

$$\psi = \alpha \frac{N_E^2}{N_\Omega}$$

where  $\alpha$  is size-independent transmission rate,  $N_E = |E| = |B \cap \Omega|$ , and  $N_\Omega = |\Omega|$ .

*Proof:* To avoid projections to the future, let us investigate what happens when a network suddenly disconnects part of its nodes:  $N_E = N_\Omega/x$  (1). The disconnected nodes try to contact the network at a constant rate  $\alpha$  and receive errors. Remaining nodes also contact old address space, therefore their success rate is proportional to remaining fraction of the network:  $\psi = \alpha \frac{N_E}{x}$  (2). From 1,  $x = \frac{N_\Omega}{N_E}$  (3). Substituting 3 into 2 we get  $\psi = \frac{\alpha N_E^2}{N_\Omega}$ . The cause of non-serviceable requests is that active information system is larger than its

accessible part. Therefore, regardless of whether network shrinks, grows, or stays constant, average fraction of satisfied demand is proportional to the square of networked fraction of the information system. ■

The next section illustrates that by example.

## VI. HETEROGENEOUS NETWORKS

Deploying a network inside of an information system implies that there exists an old way of communication between links. In fact, telecommunication networks constantly replace one another, and most networks are heterogeneous.

**Corollary 8.** *From Network Efficacy Theorem follows that when there is a default network  $D$  that connects all nodes and a preferred network  $K$  that connects part of those nodes,*

$$\psi_K + \psi_D = \frac{1}{1 - n^2}$$

where  $\psi_D, + \phi_K$  denotes the joint capacity of default and preferred network, and  $0 \leq n < 1$  is the fraction of nodes that the preferred network connects.

Let me illustrate above notion by example: Parallel computation cluster nodes produce synchronous traffic at a constant rate. All traffic is produced and consumed locally; it is evenly distributed among nodes. Nodes are connected by a network switch with total flow cap of 1 Tb/s. An additional, separate network with flow capacity of 2 Tb/s is being deployed. The task is to fully load both of networks. When two networks depend on each other, total traffic is limited to  $\frac{1}{1-n}$  of the smaller capacity, where  $n$  is fraction of traffic that goes over the faster network. If faster network took  $\frac{2}{3}$  of the load from the slower one, total capacity would be 3 Tb/s. As far, workers have installed faster network on  $\frac{2}{3}$  of the nodes. What is the overall capacity of the network?

**A:** Efficacy of the faster network  $\psi_F = \frac{(2/3N_\Omega)^2}{N_\Omega} = \frac{4}{9}$  of the slower network. As was already shown, it happens because nodes connected by the faster network can find a destination inside it only  $\frac{2}{3}$  of the time, therefore  $\frac{1}{3}$  of time they default to the slower network. As a result, total network capacity is  $1/(1 - n^2) = 1.8$  Tb/s. To achieve 3 Tb/s, workers need to connect  $n = \sqrt{1 - \frac{1}{\psi_D + \psi_K}} \approx 81.65\%$  of nodes.

## VII. SATURATED BEHAVIOR

When network size equals to information system size, adding new nodes results in equal growth of  $N_E$  and  $N_\Omega$ . As a result,  $\zeta$  stays constant. See figure 2.

Though each node accesses increased address space, each address has proportionally less value. The same applies to the amount of obtainable information.

## VIII. MULTIPURPOSE NETWORKS

A multipurpose network is one that serves many information systems. Total efficacy of a multipurpose network is the sum of its capacities for all systems. Therefore, overall size of a network may serve as a rough approximation of its efficacy.

$$\psi_{mp}^{tot} = \sum_i \psi_i \propto N^2$$

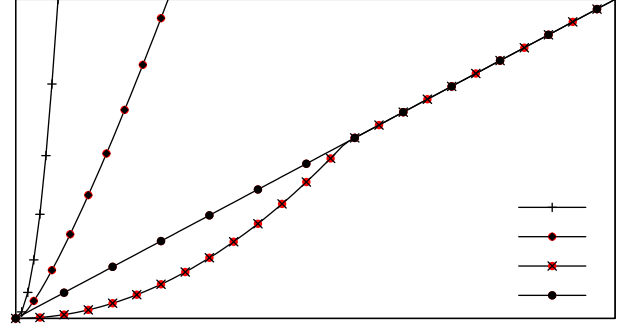


Figure 2. After a network catches the size of its underlying information system, it switches to linear growth

## IX. DISCUSSION

Studied models did not confirm moderate exponential growth estimates suggested by [2]. Their analysis also invalidates a notion that network effect can lead to exponential growth of utility. Instead, it suggests that when a network grows by extending existing or creating new information systems, its value grows linearly, regardless of the number of spawned systems. One explanation is that simple network model is just a distorted approximation of a distributed information system, and as such it has dynamics that is common to information systems.

However in a more complex setup, when a network grows within a larger information system, its efficacy raises exponentially until it catches the size of the underlying information system. That exponential growth exactly follows Metcalfe's law. Efficacy dramatically affects network utility, and may play a major role in limiting network value in heterogeneous environments.

In addition to linear growth of information, model suggests linear growth of redundancy. As information density is a reciprocal of data redundancy, network effect dictates that unique content of overall network traffic is a reciprocal of network size. However the later effect may be mitigated by limited applicability of network model to real-world information systems. It can be argued that information systems built with network model in mind, such as WWW, exhibit more redundancy than setups based on complementary subsystems. That suggests promotion of node complementarity as a possible way to reduce redundancy.

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